

Tilburg University

The egalitarian solution for convex games

Klijn, F.; Slikker, M.; Tijs, S.H.; Zarzuelo, I.

Published in:
Mathematical Social Sciences

Publication date:
2000

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
Klijn, F., Slikker, M., Tijs, S. H., & Zarzuelo, I. (2000). The egalitarian solution for convex games: Some characterizations. *Mathematical Social Sciences*, 40(1), 111-121.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.



ELSEVIER

Mathematical Social Sciences 40 (2000) 111–121

mathematical
social
sciences

www.elsevier.nl/locate/econbase

The egalitarian solution for convex games: some characterizations

Flip Klijn^{a,*}, Marco Slikker^a, Stef Tijs^a, José Zarzuelo^b

^a*Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands*

^b*Department of Applied Mathematics, University of País Vasco, Bilbao, Spain*

Received 1 August 1998; received in revised form 1 March 1999; accepted 1 June 1999

Abstract

The egalitarian solution for TU-games as introduced by Dutta and Ray [Dutta, B., Ray, D., 1989. A concept of egalitarianism under participation constraints, *Econometrica* 57, 615–635] is studied. Five characterizations of the restriction of this solution to the class of convex games are given. They all involve a stability property due to the concept of the equal division core from Selten [Selten, R., 1972. Equal share analysis of characteristic function experiments. In: Sauermann, H. (Ed.), *Beiträge zur experimentellen Wirtschaftsforschung*, Vol. 3, J.C.B. Mohr, Tübingen, pp. 130–165] and all but the third characterization involve a property restricting maximum payoffs. The first two characterizations use in addition efficiency and the reduced game properties of Hart and Mas-Colell [Hart, S., Mas-Colell, A., 1989. Potential, value and consistency. *Econometrica* 57, 589–614] and Davis and Maschler [Davis, M., Maschler, M., 1965. The kernel of a cooperative game. *Naval Research Logistics Quarterly* 12, 223–259], respectively. The fourth and fifth characterization only need in addition weak variants of the reduced game properties mentioned above. The third characterization involves besides the stability condition, efficiency and a new consistency property. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Egalitarian solution; Convex games; Characterizations

JEL classification: C71

1. Introduction

Dutta and Ray (1989) introduced the egalitarian solution as a solution concept for TU-games. This solution unifies the two conflicting concepts of individualistic utility

*Corresponding author. Tel.: +31-13-466-2733; fax: +31-13-466-3280.

E-mail address: f.klijn@kub.nl (F. Klijn)

maximization and the social goal of equality. Under certain conditions it is non-empty, and then its outcome is unique, namely it is the Lorenz maximal element of the set of payoffs satisfying core-like participation constraints. If the egalitarian solution exists, then every other feasible allocation for the grand coalition is either blocked, or is Lorenz dominated by some allocation which, in turn, is not blocked. We refer to Dutta and Ray (1989) for the details. For convex games Dutta and Ray (1989) describe an algorithm to locate the unique egalitarian solution, and they show, in addition, that it is in the core. Nevertheless, for balanced games, which have a non-empty core, the egalitarian solution might not even exist.

Dutta and Ray (1991) consider a parallel concept, the S -constrained egalitarian solution, which is not a singleton in general. The construction of the S -constrained egalitarian solution is identical to the egalitarian solution mentioned above, except that in the concept of blocking it is required that every member of the blocking coalition is strictly better off rather than at least one member, as is the case for the original egalitarian solution. They show that in contrast to the original egalitarian solution, S -constrained egalitarian solutions exist under very mild conditions on the game. The two solutions are not completely unrelated, since, for example, for convex games either the egalitarian solution is the unique S -constrained egalitarian allocation or every S -constrained egalitarian allocation Lorenz-dominates the egalitarian solution.

Dutta (1990) characterizes the egalitarian solution over the class of convex games. The properties used are the reduced game properties due to Hart and Mas-Colell (1989) and Davis and Maschler (1965). The egalitarian solution is the only solution concept satisfying either of the two reduced game properties and constrained egalitarianism, which is a prescriptive property on two person games.

Arin and Iñarra (1997) introduce another solution concept that embodies a 'bilaterally egalitarian' notion. Their solution concept is called the egalitarian set. They characterize this multi-valued solution over the class of all TU-games by extended constrained egalitarianism, the Davis-Maschler reduced game property, and the converse Davis-Maschler reduced game property. Moreover, they prove that it is non-empty for the class of balanced games. They show in addition that there is nevertheless no solution on the class of balanced games satisfying non-emptiness, extended constrained egalitarianism, and the Hart and Mas-Colell reduced game property. Arin and Iñarra (1997) show that in general the egalitarian solution of Dutta and Ray (1989) does not satisfy the Davis-Maschler reduced game property, nor the Hart and Mas-Colell reduced game property. Finally, Arin and Iñarra (1997) show that the egalitarian solution belongs to the egalitarian set, and that for convex games it even coincides with the latter.

Given the results above that for the class of convex games the egalitarian solution exists and coincides with the egalitarian set, we would like to reconsider the egalitarian solution on the class of convex games.

In this paper we provide five characterizations of the egalitarian solution, without directly making use of Dutta's (1990) prescriptive property constrained egalitarianism on two-person games. All five characterizations involve a stability property due to the concept of the equal division core from Selten (1972) and all but the third characterization involve a property restricting maximum payoffs. The first two characterizations use in addition efficiency and a reduced game property, more specifically, the reduced game

property of Hart and Mas-Colell (1989) and the reduced game property of Davis and Maschler (1965), respectively. The fourth and fifth characterization only need in addition weak variants of the reduced game properties mentioned above. Here, weak means that we only look at the reduced game where the players receiving most have been sent away. The third characterization uses besides the stability property due to the concept of the equal division core from Selten (1972) efficiency and another consistency property.

The stability property due to the concept of the equal division core from Selten (1972), called equal division stability, states that for any convex game and for any coalition there is some player in this coalition that gets at least the average of the value of the coalition in the game. Clearly, any core-allocation satisfies this property. The intuitive reasoning behind this property is spelled out in Selten (1972). The principle of equal division is a strong distributive norm which influences the behavior of the players. The attention of the players is attracted by coalitions with high equal shares. This is confirmed by the great number of cases of experimental games in Selten (1972) in which the outcome is such that there is no coalition that can divide its value equally among its members giving all of them more than in the original outcome.

The second property, concerning the boundedness of the payoffs, states that the payoffs of the players receiving most is bounded by imposing the condition that the sum of payoffs of these players does not exceed the value of the players in the game. This might be desirable from a social point of view.

The work is organized as follows. Section 2 deals with notation and definitions regarding TU-games and recalls the egalitarian solution for convex games. In Section 3 we provide several characterizations of this solution concept.

2. Preliminaries

A cooperative game with transferable utilities (TU-game) is a pair (N, v) , where $N = \{1, \dots, n\}$ is the player set and v the characteristic function, which assigns to every subset¹ S of N a value $v(S)$, with $v(\emptyset) = 0$. A game (N, v) is called *convex* if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \quad \text{for all } S, T \subseteq N.$$

The *core* of a game (N, v) is defined by

$$C(N, v) := \{x \in \mathbb{R}^N : x(N) = v(N) \quad \text{and} \quad x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

Throughout this paper we will denote the *average worth* of coalition S in game (N, v) by

¹ $S \subseteq N$ denotes that S is a subset of N and $S \subset N$ denotes that S is a strict subset of N .

$$a(S, v) := \frac{v(S)}{|S|}.$$

We will recall the algorithm of Dutta and Ray (1989) to locate the egalitarian solution for convex games. In every step of the algorithm a cooperative game is considered. The set of players in this game is the set of players that have not received a payoff yet. The largest coalition with the highest average worth is selected and the players in this coalition receive this average worth.

Let (N, v) be a convex TU-game. Define $N_1 := N$ and $v_1 := v$.

Step 1. Let S_1 be the largest coalition with the highest average worth in the game (N_1, v_1) . Define

$$E_i(N, v) := a(S_1, v_1) \quad \text{for all } i \in S_1.$$

Step k . Suppose that S_1, \dots, S_{k-1} have been defined recursively and $S_1 \cup \dots \cup S_{k-1} \neq N$. Define a new game with player set $N_k := N_{k-1} \setminus S_{k-1} = N \setminus (S_1 \cup \dots \cup S_{k-1})$. For all subcoalitions $S \subseteq N_k$, define $v_k(S) := v_{k-1}(S_{k-1} \cup S) - v_{k-1}(S_{k-1})$. Convexity of (N_{k-1}, v_{k-1}) implies convexity of (N_k, v_k) . Define S_k to be the largest coalition with the highest average worth in this game. Define

$$E_i(N, v) := a(S_k, v_k) \quad \text{for all } i \in S_k.$$

It can be checked that in every step convexity ensures the existence of a largest coalition with highest average worth. In at most n steps the algorithm ends, and the constructed allocation $E(N, v)$ is called the egalitarian solution of the game (N, v) . Dutta and Ray (1989) show that $E(N, v)$ is an element of the core of (N, v) . Furthermore, they note that for each convex game (N, v) it holds that

$$E_i(N, v) > E_j(N, v), \quad \text{for all } i \in S_k, j \in S_{k+1}. \quad (1)$$

3. Characterizations of the egalitarian solution

In this section we provide several characterizations of the egalitarian solution for convex games. The first two characterizations are based on characterizations of Dutta (1990) in the sense that we replace a prescriptive property on two-person games by some other properties. The third characterization uses a new consistency property, whereas the last two characterizations strengthen the first two characterizations by weakening the respective consistency properties and not demanding efficiency a priori.

Our first characterization of the egalitarian solution for convex TU-games involves the properties efficiency, equal division stability, bounded maximum payoff property, and HM consistency. We describe these properties below. Let \mathcal{C} be the set of convex TU-games. A solution on \mathcal{C} is a map ψ assigning to each convex game $(N, v) \in \mathcal{C}$ an element $\psi(N, v) \in \mathbb{R}^N$. Let (N, v) be a convex game. Given the solution ψ , define $S^m(N,$

v, ψ) (or S^m for short, if no confusion is possible) to be the set of players with the highest payoff. Formally,

$$S^m = S^m(N, v, \psi) := \operatorname{argmax}_{j \in N} \psi_j(N, v).$$

A solution ψ on \mathcal{C} satisfies

- *efficiency* (EFF) if for all games $(N, v) \in \mathcal{C}$:

$$\sum_{i \in N} \psi_i(N, v) = v(N).$$

- *equal division stability* (EDS) if for all games $(N, v) \in \mathcal{C}$ and all $S \subseteq N, S \neq \emptyset$ there exists $i \in S$ with

$$\psi_i(N, v) \geq a(S, v).$$

- *bounded maximum payoff property* (BMPP) if for all games $(N, v) \in \mathcal{C}$:

$$\sum_{i \in S^m} \psi_i(N, v) \leq v(S^m).$$

- *HM consistency* (HMC) if for all games $(N, v) \in \mathcal{C}$, all $S \subseteq N$, and all $i \in N \setminus S$:

$$\psi_i(N, v) = \psi_i(N \setminus S, v^{-S}),$$

where v^{-S} is the reduced subgame defined by

$$v^{-S}(T) := v(S \cup T) - \sum_{i \in S} \psi_i(S \cup T, v)$$

for all subcoalitions $T \subseteq N \setminus S$.

- *constrained egalitarianism* (CE) if for all games $(\{i, j\}, v) \in \mathcal{C}$:

$$\psi_i(\{i, j\}, v) = \begin{cases} \max \left(v(\{i\}), \frac{v(\{i, j\})}{2} \right) & \text{if } v(\{j\}) \leq \frac{v(\{i, j\})}{2}; \\ v(\{i, j\}) - v(\{i\}) & \text{otherwise.} \end{cases}$$

(EDS) plays a role in the concept of equal division core from Selten (1972). (BMPP) states that the payoffs of the players receiving most is bounded, which might be desirable from a social point of view. (HMC) is the well-known consistency property of Hart and Mas-Colell (1989). Dutta (1990) characterizes the egalitarian solution with (HMC) and (CE).

Theorem 3.1 shows that we can replace the prescriptive property (CE) by the properties (EFF), (EDS), and (BMPP). But first we prove two lemmas.

Lemma 3.1. *If a solution ψ satisfies (EDS) and (BMPP) then for all $(N, v) \in \mathcal{C}$ and all $i \in S^m$*

$$\psi_i(N, v) = a(S^m, v).$$

Proof. Let ψ be a solution that satisfies (EDS) and (BMPP). Let (N, v) be a convex

game. Suppose there is a player $i \in S^m$ such that $\psi_i(N, v) < a(S^m, v)$. Since all players in S^m receive the same payoff we have a contradiction with (EDS). Hence, $\psi_i(N, v) \geq a(S^m, v)$ for all $i \in S^m$. By (BMPP), it then immediately follows that $\psi_i(N, v) = a(S^m, v)$ for all $i \in S^m$. \square

Lemma 3.2. *If a solution ψ satisfies (EFF), (EDS), and (BMPP) then it also satisfies (CE).*

Proof. Let ψ be a solution that satisfies (EFF), (EDS), and (BMPP). Let $(\{1, 2\}, v)$ be a convex game. Suppose without loss of generality that

$$\psi_1(\{1, 2\}, v) \geq \psi_2(\{1, 2\}, v). \quad (2)$$

Suppose the inequality in Eq. (2) is an equality. Then

$$\psi_1(\{1, 2\}, v) = \psi_2(\{1, 2\}, v) = \frac{v(\{1, 2\})}{2}$$

by Lemma 3.1. So, together with (EDS) it follows that

$$\frac{v(\{1, 2\})}{2} = \psi_i(\{1, 2\}, v) \geq v(\{i\}).$$

One easily verifies that ψ satisfies the condition in (CE).

Now suppose the inequality in Eq. (2) is strict. Then, by Lemma 3.1, $\psi_1(\{1, 2\}, v) = v(\{1\})$. So, with (EFF), $\psi_2(\{1, 2\}, v) = v(\{1, 2\}) - v(\{1\})$. Hence,

$$v(\{1\}) = \psi_1(\{1, 2\}, v) > \psi_2(\{1, 2\}, v) = v(\{1, 2\}) - v(\{1\}).$$

So,

$$v(\{1\}) > \frac{v(\{1, 2\})}{2}.$$

Furthermore, by convexity,

$$v(\{1, 2\}) \geq v(\{1\}) + v(\{2\}) > \frac{v(\{1, 2\})}{2} + v(\{2\}).$$

Hence,

$$v(\{2\}) < \frac{v(\{1, 2\})}{2}.$$

Then again it is readily verified that ψ satisfies the condition in (CE). \square

We provide our first characterization in the following theorem.

Theorem 3.1. *A solution ψ satisfies (EFF), (EDS), (BMPP), and (HMC) if and only if $\psi = E$.*

Proof. First we show that E satisfies the properties.

Since E assigns to every convex game a core element, it satisfies (EFF) and (EDS). It follows from Eq. (1) that every player in S_1 receives the maximum payoff and that all other players receive less than this maximum. Since these players divide $v(S_1)$ it follows that E satisfies (BMPP). From Dutta (1990) it follows that E satisfies the reduced game property of Hart and Mas-Colell (1989).

Suppose a solution ψ satisfies the four properties in the theorem. Then, by Lemma 3.2 ψ also satisfies (CE). Then, by Dutta's (1990) characterization with (CE) and (HMC) it immediately follows that $\psi = E$. \square

The previous characterization still holds true if we replace the consistency property of Hart and Mas-Colell (1989) with the consistency property of Davis and Maschler (1965). A solution ψ on \mathcal{C} satisfies

- DM consistency (DMC) if for all games $(N, v) \in \mathcal{C}$, all $S \subseteq N$, and all $i \in N \setminus S$:

$$\psi_i(N, v) = \psi_i(N \setminus S, v_{-S}),$$

where v_{-S} is the reduced subgame defined by

$$v_{-S}(T) := \begin{cases} 0 & \text{if } T = \emptyset; \\ v(N) - \sum_{i \in S} \psi_i(N, v) & \text{if } T = N \setminus S; \\ \max_{Q \subseteq S} \left\{ v(T \cup Q) - \sum_{i \in Q} \psi_i(N, v) \right\} & \text{if } \emptyset \subset T \subset N \setminus S. \end{cases}$$

Theorem 3.2. A solution ψ satisfies (EFF), (EDS), (BMPP), and (DMC) if and only if $\psi = E$.

Proof. This immediately follows from Lemma 3.2 and Dutta's (1990) characterization with (DMC). \square

Another characterization of the egalitarian solution is obtained when we use the next, third consistency property. This property only puts a condition on the reduced game where the players receiving most are sent away. A solution ψ on \mathcal{C} satisfies

- max-consistency (MC) if for all games $(N, v) \in \mathcal{C}$ and all $i \in N \setminus S^m$:

$$\psi_i(N, v) = \psi_i(N \setminus S^m, v^{-S^m}),$$

where v^{-S^m} is the reduced subgame defined by

$$v^{-S^m}(T) := v(S^m \cup T) - v(S^m)$$

for all subcoalitions $T \subseteq N \setminus S^m$.

Before we prove the characterization we prove another lemma.

Lemma 3.3. *If a solution ψ satisfies (EFF) and (MC) then for all $(N, v) \in \mathcal{C}$ and all $i \in S^m$*

$$\psi_i(N, v) = a(S^m, v).$$

Proof. Let ψ be a solution that satisfies (EFF) and (MC). Let (N, v) be a convex game. By efficiency of $\psi(N, v)$ and $\psi(N \setminus S^m, v^{-S^m})$ it follows using (MC) that $\sum_{i \in S^m} \psi_i(N, v) = v(S^m)$. By definition of S^m it follows that for all $i \in S^m$ it holds that $\psi_i(N, v) = a(S^m, v)$. \square

We can now prove the third characterization.

Theorem 3.3. *A solution ψ satisfies (EFF), (EDS), and (MC) if and only if $\psi = E$.*

Proof. Since from the algorithm of the egalitarian solution it immediately follows that E satisfies (MC), we prove the “only if”-part. Suppose that a solution ψ satisfies the properties. We prove that $\psi = E$. The proof will be by induction on the number of players.

Clearly, for convex games (N, v) with $|N| = 1$ we have that $\psi(N, v) = v(\{1\}) = E(N, v)$ by Lemma 3.3. Suppose that for some $p \geq 2$ we have $\psi(N, v) = E(N, v)$ for all convex games (N, v) with $|N| \leq p - 1$. We prove that $\psi(N, v) = E(N, v)$ also holds for all convex games (N, v) with $|N| = p$.

Let (N, v) be a convex game with $|N| = p$. Let S_1 be the largest coalition that maximizes the average worth function $a(\cdot, v)$. First we will show that $a(S_1, v) = a(S^m, v)$. Since ψ satisfies (EDS) there exist $i \in S_1$ with $\psi_i(N, v) \geq a(S_1, v)$. Then for all $j \in S^m$ we have

$$a(S^m, v) = \psi_j(N, v) \geq \psi_i(N, v) \geq a(S_1, v),$$

where the equality follows from Lemma 3.3. The first inequality follows by definition of S^m . Since the definition of S_1 implies $a(S_1, v) \geq a(S^m, v)$ we conclude

$$a(S^m, v) = a(S_1, v). \quad (3)$$

Again by definition of S_1 this implies $S^m \subseteq S_1$. We will show that $S^m = S_1$.

Suppose $S^m \subset S_1$. With $T = S_1 \setminus S^m \neq \emptyset$ (MC) gives

$$v^{-S^m}(S_1 \setminus S^m) = v(S_1) - v(S^m),$$

because $S^m \subseteq S_1$. But then

$$\frac{v(S_1)}{|S_1|} = \frac{v(S_1) - v(S^m) + v(S^m)}{|S_1 \setminus S^m| + |S^m|} = \frac{v^{-S^m}(S_1 \setminus S^m) + v(S^m)}{|S_1 \setminus S^m| + |S^m|}.$$

From this and (3) it follows that

$$a(S_1, v) = \frac{v(S_1)}{|S_1|} = \frac{v^{-S^m}(S_1 \setminus S^m)}{|S_1 \setminus S^m|}. \quad (4)$$

Now, using the convexity of the reduced game $(N \setminus S^m, v^{-S^m})$ it follows that

$$\psi_i(N \setminus S^m, v^{-S^m}) = \psi_i(N, v) < a(S_1, v) = \frac{v^{-S^m}(S_1 \setminus S^m)}{|S_1 \setminus S^m|} \quad \text{for all } i \in S_1 \setminus S^m, \quad (5)$$

where the first equality follows from (MC), the strict inequality from Eq. (3) and the definition of S^m , and the second equality from Eq. (4). Inequality Eq. (5) contradicts with (EDS) of $S_1 \setminus S^m$ in the reduced game $(N \setminus S^m, v^{-S^m})$. Hence, the assumption $S^m \subset S_1$ is false. Since $S^m \subseteq S_1$, this completes the proof of $S^m = S_1$.

It remains to prove that indeed from $S^m = S_1$ it follows that $\psi(N, v) = E(N, v)$. Note first that Lemma 3.3, the definition of S^m , and $S^m = S_1$ yield

$$\psi_i(N, v) = \frac{v(S^m)}{|S^m|} = E_i(N, v) \quad \text{for all } i \in S^m. \quad (6)$$

Then, if $S^m = N$ we are done. If $S^m \neq N$ it holds that

$$\psi_i(N, v) = \psi_i(N \setminus S^m, v^{-S^m}) = E_i(N \setminus S^m, v^{-S^m}) = E_i(N, v) \quad \text{for all } i \in N \setminus S^m, \quad (7)$$

where the first equality follows from (MC), the second equality from the induction hypothesis, and the third equality from $S^m = S_1$. The theorem now follows from Eq. (6) and Eq. (7). \square

It follows from the following examples that the properties in Theorem 3.3 are logically independent.

- The solution that equally divides the worth of the grand coalition to the players satisfies (EFF) and (MC), but does not satisfy (EDS).
- The solution $\alpha(N, v) = 2E(N, v)$ satisfies (EDS) and (MC), but not (EFF).
- The Shapley value satisfies (EFF) and (EDS), but not (MC).

Property (EFF) in Theorem 3.3 can be replaced by (BMPP). The proof then follows the lines of the proof of Theorem 3.3 replacing Lemma 3.3 by Lemma 3.1.

A fourth and fifth characterization are obtained by (EDS), (BMPP), together with either a weaker variant of the reduced game property of Hart and Mas-Colell (1989) or a weaker variant of the reduced game property of Davis and Maschler (1965). Formally, a solution ψ on \mathcal{C} satisfies

- *HM max-consistency* (HMMC) if for all games $(N, v) \in \mathcal{C}$, and all $i \in N \setminus S^m$:

$$\psi_i(N, v) = \psi_i(N \setminus S^m, v^{-S^m}),$$

where v^{-S^m} is the reduced subgame defined by

$$v^{-S^m}(T) := v(S^m \cup T) - \sum_{i \in S^m} \psi_i(S^m \cup T, v)$$

for all subcoalitions $T \subseteq N \setminus S^m$.

DM max-consistency (DMMC) is defined similar to (HMMC) but with the reduced game defined in property (DMC). So, (HMMC) and (DMMC) are weak variants of (HMC) and (DMC), respectively. This restriction is quite natural since the players receiving most are the players that might have the greatest incentive to walk away from the grand coalition.

Thus, our fourth and fifth characterization of the egalitarian solution are as follows.

Theorem 3.4. *A solution ψ satisfies (EDS), (BMPP), and (HMMC) if and only if $\psi = E$. A solution ψ satisfies (EDS), (BMPP), and (DMMC) if and only if $\psi = E$.*

The proofs of the characterizations are closely related to the proof of Theorem 3.3 and can be found in Klijn et al. (1998). The following examples show that the properties in the two characterizations of Theorem 3.4 are logically independent.

- The solution that equally divides the worth of the grand coalition to the players satisfies (BMPP), (HMMC), and (BMMC), but does not satisfy (EDS).
- The Shapley value satisfies (EDS) and (HMMC) (see Hart and Mas-Colell, 1989), but not (BMPP).
- Sobolev (1975) showed that the prenucleolus satisfies (DMMC). Furthermore, the prenucleolus belongs to the core for convex games and hence it satisfies (EDS). Finally, the prenucleolus does not satisfy (BMPP).
- We will define the solution β . Let S_1 be the coalition that is determined in the first step of the algorithm to determine E . If $S_1 \neq N$, define:

$$\bar{v}_{-S_1}(T) := \begin{cases} v(N) - v(S_1) & \text{if } T = N \setminus S_1; \\ v(T) & \text{if } T \subset N \setminus S_1. \end{cases}$$

Note that $v(N) - v(S_1) \geq v(N \setminus S_1)$. Hence, $(N \setminus S_1, \bar{v}_{-S_1})$ is convex. Now, let

$$\beta_i(N, v) := \begin{cases} E_i(N, v) & \text{if } i \in S_1; \\ E_i(N \setminus S_1, \bar{v}_{-S_1}) & \text{if } i \notin S_1. \end{cases}$$

Obviously, β satisfies (EFF). We will show that β satisfies (EDS). Let $S \subseteq N$, $S \neq \emptyset$. If $S \cap S_1 \neq \emptyset$ then with $i \in S \cap S_1$ it holds that $\beta_i(N, v) = E_i(N, v) \geq a(S, v)$. If $S \subseteq N \setminus S_1$ it follows that there exists $i \in S$ with $\beta_i(N, v) = E_i(N \setminus S_1, \bar{v}_{-S_1}) \geq a(S, \bar{v}_{-S_1}) \geq a(S, v)$, where the last inequality follows by definition of \bar{v} . So, β also satisfies (EDS). Finally we will show that $\beta \neq E$ implying that β satisfies neither (HMMC) nor (DMMC). Consider the three-person cooperative game $(\{1, 2, 3\}, v)$ defined by

$$v(S) = \begin{cases} 10 & \text{if } S \in \{\{1\}, \{1, 3\}\}; \\ 16 & \text{if } S = \{1, 2\}; \\ 20 & \text{if } S = N; \\ 0 & \text{otherwise.} \end{cases}$$

Then it can be checked that $S_1 = \{1\}$, $E(N, v) = (10, 6, 4)$, and $\beta(N, v) = (10, 5, 5)$. Hence, β satisfies neither (HMMC) nor (DMMC).

The main conclusion following from these examples is the following: considering (EFF), (EDS), (BMPP), and (HMMC)/(DMMC), then obviously, Theorem 3.4 implies that E can be characterized by these four properties. Furthermore, Theorem 3.4 implies that E can be characterized omitting (EFF), but the examples show that we cannot omit any of the other properties. Finally, note that Theorem 3.4 strengthens Theorems 3.1 and 3.2 by omitting efficiency and requiring only a weak consistency property.

Acknowledgements

The authors thank Herbert Hamers and two anonymous referees for useful suggestions and comments.

References

- Arin, J., Iñarra, E., 1997. Consistency and egalitarianism: the egalitarian set. Discussion Paper 163, University of País Vasco, Bilbao, Spain, 1997.
- Davis, M., Maschler, M., 1965. The kernel of a cooperative game. *Naval Research Logistics Quarterly* 12, 223–259.
- Dutta, B., 1990. The egalitarian solution and reduced game properties in convex games. *International Journal of Game Theory* 19, 153–169.
- Dutta, B., Ray, D., 1989. A concept of egalitarianism under participation constraints. *Econometrica* 57, 615–635.
- Dutta, B., Ray, D., 1991. Constrained egalitarian allocations. *Games and Economic Behavior* 3, 403–422.
- Hart, S., Mas-Colell, A., 1989. Potential, value and consistency. *Econometrica* 57, 589–614.
- Klijn, F., Slikker, M., Tijs, S., Zarzuelo, J., 1998. Characterizations of the egalitarian solution for convex games, CentER Discussion Paper 9833, Tilburg University, The Netherlands.
- Selten, R., 1972. Equal share analysis of characteristic function experiments. In: Sauermann, H. (Ed.), *Beiträge zur Experimentellen Wirtschaftsforschung*, Vol. 6, J.C.B. Mohr, Tübingen, pp. 130–165.
- Sobolev, A., 1975. The characterization of optimality principles in cooperative games by functional equations, In: Vorobjev, N. (Ed.), *Mathematischeskie metody v socialnix naukakh*, Proceedings of the seminar, Vilnius: Institute of Physics and Mathematics, Academy of Sciences of the Lithuanian SSR, Vol. 6, pp. 94–151 [In Russian, English summary].